GROUP-INVARIANT PROPERTIES OF A NONLINEAR OPTIMALLY CONTROLLED DISTRIBUTED-PARAMETER PROCESS

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V.G. PAVLOV and V.P. CHEPRASOV
(Kazan)

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We pose the problem of constructing the optimal control for a process described by a nonlinear equation of the same form as the heat conduction equation. On the basis of the theory developed by Ovsiannikov [1] we find transformation groups which enable us to reduce the system of partial differential equations of the problem to a system of ordinary differential equations. Several types of boundary conditions formulated to conform to the resulting transformations are considered.

1. Formulation of the problem. Let the controlled process be described by the equation in dimensionless variables

$$\frac{\partial \varphi}{\partial t} - \frac{\partial}{\partial x}\left(f(\varphi)\frac{\partial \varphi}{\partial x}\right) + \alpha\varphi - \alpha u = 0 \qquad (1.1)$$

where $\varphi(t, x)$ is the required distribution, $f(\varphi)$ is some nonlinear function whose form will be determined later, α is a constant, and $u(t, x)$ is the distributed control

$$t \in [0,T], \quad x \in [0,l]$$

An equation of the form of (1.1) describes heating processes, processes in chemical reactors, etc.

Let us find the control $u(t, x)$ which minimizes the functional

$$J_u = \int_0^T \int_0^l Q u^2 dx dt \qquad (1.2)$$

where Q, T, and l are known positive constants. The boundary conditions for Eq. (1.1) will be formulated below in conformance to the resulting transformations. We shall then attempt to find transformation groups which will enable us to find group-invariant solutions of unit rank.

It is convenient to represent Eq. (1.1) in the form of the system

$$\frac{\partial \varphi}{\partial t} - \frac{\partial w}{\partial x} + \alpha\varphi - \alpha u = 0, \qquad w - f(\varphi)\frac{\partial \varphi}{\partial x} = 0 \qquad (1.3)$$

In order to solve the variational problem we make use of the Lagrange formalism which yields the necessary conditions for the extremum of (1.2) in the form of Ostrogradskii equations,

$$\lambda_1(t, x)\alpha - \lambda_2(t, x)\frac{df}{d\varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x}[\lambda_2(t, x) f(\varphi)] - \frac{\partial \lambda_1}{\partial t} = 0 \qquad (1.4)$$

$$\lambda_2 + \frac{\partial \lambda_1}{\partial x} = 0, \qquad 2Qu - \alpha\lambda_1 = 0 \qquad (1.5)$$

From Eqs. (1.5) we find that the Lagrange multipliers are given by

$$\lambda_1 = \frac{2Q}{\alpha} u, \qquad \lambda_2 = -\frac{2Q}{\alpha}\frac{\partial u}{\partial x}$$

Eq. (1.4) becomes

$$\frac{\partial u}{\partial t} + f(\varphi)\frac{\partial^2 u}{\partial x^2} - \alpha u = 0 \tag{1.6}$$

The equations of the variational problem can now be written as

$$(S) \qquad \varphi_t = w_x - \alpha\varphi + \alpha u, \qquad \varphi_x = w\,[f(\varphi)]^{-1} \tag{1.7}$$

$$u_t = \alpha u - f(\varphi)\,v_x, \quad u_x = v \tag{1.8}$$

2. Construction of the basic group G. A system of equations (S) includes the four required functions φ, u, w, v of the two independent variables x and t. The totality of the dependent and independent quantities can be regarded as the collection of coordinates of a point in the space E_6.

The transformation group G is defined by a Lie algebra of infinitesimal operators,

$$Y = \xi_t\frac{\partial}{\partial t} + \xi_x\frac{\partial}{\partial x} + \xi_\varphi\frac{\partial}{\partial\varphi} + \xi_u\frac{\partial}{\partial u} + \xi_w\frac{\partial}{\partial w} + \xi_v\frac{\partial}{\partial v} \tag{2.1}$$

where ξ_t, ξ_x, ξ_φ, ξ_u, ξ_w, ξ_v are the coordinates of the operator Y which are functions of the coordinates of the space E_6.

We now introduce the space E_{14} which is an extension of the space E_6. A point in E_{14} is defined by the coordinates

$$t,\ x,\ \varphi,\ u,\ w,\ v,\ \varphi_t,\ \varphi_x,\ u_t,\ u_x,\ w_t,\ t_x,\ x_t,\ v_x$$

Next, we introduce the group G^*, which is the first extension of the transformation group G. The group G^* is isomorphic to G. The operator of the group G^* is given by Expression (2.2)

$$Y^* = Y + \xi_{\varphi_t}\frac{\partial}{\partial\varphi_t} + \xi_{\varphi_x}\frac{\partial}{\partial\varphi_x} + \xi_{u_t}\frac{\partial}{\partial u_t} + \xi_{u_x}\frac{\partial}{\partial u_x} + \xi_{w_t}\frac{\partial}{\partial w_t} + \xi_{w_x}\frac{\partial}{\partial w_x} + \xi_{v_t}\frac{\partial}{\partial v_t} + \xi_{v_x}\frac{\partial}{\partial v_x}$$

The group G is basic to (S) if and only if the invariance conditions [1]

$$Y^*[\Psi] = 0 \tag{2.3}$$

where Ψ is a manifold defined by (S), are fulfilled in E_{14}. Conditions (2.3) then become

$$\xi_{\varphi_t} - \xi_{w_x} + \alpha\xi_\varphi - \alpha\xi_u = 0, \quad f(\varphi)\,\xi_{\varphi_x} + \frac{df}{d\varphi}\varphi_x\xi_\varphi - \xi_w = 0$$

$$\xi_{u_t} + f(\varphi)\,\xi_{v_x} + \frac{df}{d\varphi}v_x\xi_\varphi - \alpha\xi_u = 0, \quad \xi_{u_x} - \xi_v = 0 \tag{2.4}$$

The expressions for the coordinates of the extended operator in terms of the coordinates of the operator Y and the coordinates of the space E_{14} can be obtained as described in [1]; for example,

$$\xi_{\varphi_t} = D_t(\xi_\varphi) - \varphi_t D_t(\xi_t) - \varphi_x D_t(\xi_x), \qquad \xi_{\varphi_x} = D_x(\xi_\varphi) - \varphi_t D_x(\xi_t) - \varphi_x D_x(\xi_x)$$

$$D_t = \frac{\partial}{\partial t} + \varphi_t\frac{\partial}{\partial\varphi} + u_t\frac{\partial}{\partial u} + w_t\frac{\partial}{\partial w} + v_t\frac{\partial}{\partial v}$$

$$D_x = \frac{\partial}{\partial x} + \varphi_x\frac{\partial}{\partial\varphi} + u_x\frac{\partial}{\partial u} + w_x\frac{\partial}{\partial w} + v_x\frac{\partial}{\partial v}$$

The expressions for the remaining coordinates can be written out in the same way.

The invariance conditions enable us to obtain the system of defining equations of the Lie algebra. The unknown functions in this system are the coordinates of the operator Y; the independent variables are t, x, φ, u, w, v.

Study of the defining equations for the coordinates of infinitesimal operator (2.1) yields the following relations:

$$\xi_\varphi = \frac{f}{f'}\left(2\frac{\partial \xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) \tag{2.5}$$

$$\xi_w = w\left\{\frac{\partial \xi_x}{\partial x}\left[1 + 2\left(\frac{f}{f'}\right)'\right] - \frac{d\xi_t}{dt}\left[1 + \left(\frac{f}{f'}\right)'\right]\right\} + 2\frac{f^2}{f'}\frac{\partial^2 \xi_x}{\partial x^2} \tag{2.6}$$

$$\begin{aligned}\xi_u = {} & \frac{f}{f'}\left(2\frac{\partial \xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) + \frac{f}{\alpha f'}\left(2\frac{\partial^2 \xi_x}{\partial x^2} - \frac{d^2\xi_t}{dt^2}\right) + \\ & + (u-\varphi)\left(\frac{f}{f'}\right)'\left(2\frac{\partial \xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) + (\varphi - u)\frac{d\xi_t}{dt} - \frac{w}{\alpha f}\frac{\partial \xi_x}{\partial t} - \frac{w}{\alpha}\frac{\partial^2 \xi_x}{\partial x^2}\left[1 + 2\left(\frac{f}{f'}\right)'\right] - \\ & - \frac{w^2}{\alpha f}\left(\frac{f}{f'}\right)''\left[2\frac{\partial \xi_x}{\partial x} - \frac{d\xi_t}{dt}\right] - \frac{2}{\alpha}\frac{f^2}{f'}\frac{\partial^3 \xi_x}{\partial x^3} - \frac{2w}{\alpha f}\left(\frac{f^2}{f'}\right)'\frac{\partial^2 \xi_x}{\partial x^2}\end{aligned} \tag{2.7}$$

$$\xi_v = v\left(\frac{\partial \xi_u}{\partial u} - \frac{\partial \xi_x}{\partial x}\right) + \frac{\partial \xi_u}{\partial x} \tag{2.8}$$

$$\alpha\xi_u - f\left(\frac{\partial \xi_v}{\partial x} + v\frac{\partial \xi_v}{\partial u}\right) - \frac{\partial \xi_u}{\partial t} - \alpha u\frac{\partial \xi_u}{\partial u} + \alpha u\frac{d\xi_t}{dt} + v\frac{\partial \xi_x}{\partial t} = 0 \tag{2.9}$$

Here and below the prime denotes differentiation with respect to φ. Moreover,

$$\xi_t = \xi_t(t), \quad \xi_x = \xi_x(t, x), \quad \xi_\varphi = \xi_\varphi(t, x, \varphi)$$
$$\xi_u = \xi_u(t, x, u), \quad \xi_w = \xi_w(t, x, \varphi, w), \quad \xi_v = \xi_v(t, x, u, v)$$

We can now use the resulting relations (2.5)-(2.9) to investigate the group properties of system (S) for certain types of linear functions $f(\varphi)$ and boundary conditions.

A. Let us consider the case of determining the basic group for an arbitrary function $f(\varphi)$. Since ξ_u does not depend on ϕ and w, (2.7) implies the relations

$$\frac{d\xi_t}{dt} = \frac{\partial \xi_x}{\partial t} = \frac{\partial \xi_x}{\partial x} = \frac{\partial^2 \xi_x}{\partial x^2} = \frac{\partial^3 \xi_x}{\partial x^3} = 0$$

Eq. (2.9) is satisfied identically. Here the coordinates ξ_t and ξ_x of the operator are the defining constants, and $\xi_\varphi = \xi_w = \xi_u = \xi_v = 0$. Hence, the basis of the Lie algebra of the basic group of system (S) consists of the operators

$$Y_1 = \frac{\partial}{\partial t} \qquad Y_2 = \frac{\partial}{\partial x}$$

The representatives of the classes of similar subalgebras of unit order are then of the form [1]

$$\langle Y_2\rangle, \qquad \langle Y_1 + KY_2\rangle \tag{2.10}$$

where K is any real number. These classes are associated with the subgroups H_1 and H_2 of the basic group G.

B. The transformation group G can be extended by means of a special form of the function $f(\varphi)$. It is clear from (2.7) that ξ_u does not depend on w and φ for

$$f(\varphi) = C_1\varphi^{2m} \quad \text{or} \quad f(\varphi) = C_2e^{n\varphi} \tag{2.11}$$

where C_1, C_2, m and n are arbitrary constants, and

$$\frac{d\xi_t}{dt} = \frac{\partial^2 \xi_x}{\partial x^2} = \frac{\partial^3 \xi_x}{\partial x^3} = \frac{\partial \xi_x}{\partial t} = 0$$

In this case the coordinates of the infinitesimal operator Y are

$$\xi_t = \xi_{t0}, \quad \xi_x = \xi_{x0} + \xi_{x1}x, \quad \xi_\varphi = m^{-1}\xi_{x1}\varphi, \quad \xi_u = m^1\xi_{x1}u \tag{2.12}$$
$$\xi_w\,(m+1)m^{-1}\xi_{x1}w, \quad \xi_v = (1-m)m^{-1}\xi_{x1}v$$

The defining constants are ξ_{t0}, ξ_{x0}, ξ_{x1}.
The representatives of the classes of similar subalgebras of unit order are now

$$\langle Y_2\rangle,\ \langle Y_1+K_1Y_2\rangle,\ \langle K_2Y_1+Y_3\rangle$$
$$Y_1=\frac{\partial}{\partial t},\qquad Y_2=\frac{\partial}{\partial x} \tag{2.13}$$
$$Y_3=x\frac{\partial}{\partial x}+\frac{\varphi}{m}\frac{\partial}{\partial\varphi}+\frac{u}{m}\frac{\partial}{\partial u}+\frac{m+1}{m}w\frac{\partial}{\partial w}+\frac{1-m}{m}v\frac{\partial}{\partial v}$$

Here K_1 and K_2 are arbitrary real numbers.
These classes are associated with the subgroups H_1', H_2, H_3 of the basic group G.

3. Group-invariant solutions of system (S). A complete collection of functionally independent invariants I_j $(j=1,2,\ldots,5)$ can be found for each subgroup.

If the manifold defined by the equations $\varphi=\varphi(x,t)$, $u=u(x,t)$, $w=w(x,t)$, $v=v(x,t)$, is nonsingular, i.e. if the rank of the matrix of the coordinates of the infinitesimal operators at the points of the manifold is not smaller than the total rank of this matrix, then the manifold is defined by the system of equations

$$\Phi^{\beta}(I_1,I_2,\ldots)=0\qquad(\beta=1,\ldots,4)$$

Taking any invariant I_j as our new independent variable, we can find the relations $I_k(I_j)$ $(k\neq j)$. Solving these for φ, u, w, v and substituting these into (S), we obtain a system of ordinary differential equations which we denote by S/H_i $(i=1,2,3)$.

Let us now consider the group-invariant solutions for Cases A and B. We have shown that each transformation subgroup H_i must be matched by boundary conditions of a certain type which cannot be formulated until the transformation has been determined.

A°. The complete collection of functionally independent invariants for a representative of the class $\langle Y_2\rangle$ is of the form

$$I_1=t,\quad I_2=\varphi(t),\quad I_3=u(t)$$

The system S/H_1 can be written as

$$\frac{dI_2}{dI_1}=\alpha(I_3-I_2),\qquad\frac{dI_3}{dI_1}=\alpha I_3 \tag{3.1}$$

The resulting transformation subgroup can be used if the boundary conditions for the initial problem are specified (for example) in the form

$$\varphi(x,0)=\varphi_0,\qquad u(x,0)=u_0,\qquad\varphi(x,T)=\varphi_T$$

Here φ_0, u_0, φ_T are constants. The equation $\varphi(x,T)=\varphi_T$ serves to define the instant T of termination of the controlled process.

The subgroup H_2 with the operator Y_1+KY_2 is associated with the invariants

$$I_1=x-kt,\ I_2=\varphi(x,t),\ J_3=u(x,t),\ I_4=w(x,t),\ I_5=v(x,t)$$

The system S/H_2 is of the form

$$k\frac{dI_2}{dI_1}=\alpha I_2-\alpha I_3-\frac{dI_4}{dI_1},\qquad f(I_2)\frac{dI_2}{dI_1}=I_4 \tag{3.2}$$
$$k\frac{dI_3}{dI_1}=f(I_2)\frac{dI_5}{dI_1}-\alpha I_3,\qquad\frac{dI_3}{dI_1}=I_5$$

By eliminating I_4 and I_5 we reduce system (3.2) to

$$\frac{d^2I_2}{dI_1^2}=\frac{1}{f(I_2)}\left[\alpha(I_2-I_3)-\frac{dI_2}{dI_1}\left(\frac{df(I_2)}{dI_1}+k\right)\right],\qquad\frac{d^2I_3}{dI_1^2}=\frac{1}{f(I_2)}\left(k\frac{dI_3}{dI_1}+\alpha I_3\right) \tag{3.3}$$

In this case the boundary conditions can be formulated as

$$I_2(0) = \varphi(0, 0), \qquad I_3(0) = u(0, 0), \qquad \left.\frac{dI_2}{dI_1}\right|_{I_1=0} = \left.\frac{\partial\varphi}{\partial x}\right|_{\substack{x=0\\t=0}}, \qquad \left.\frac{dI_3}{dI_1}\right|_{I_1=0} = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=0\\t=0}}$$

We can determine the instant of termination of the process from the condition

$$\varphi(0, T) = \varphi_T$$

B°. This variant differs from A° in that it involves the operator

$$k_2Y_1 + Y_3 = k_2\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{\varphi}{m}\frac{\partial}{\partial\varphi} + \frac{u}{m}\frac{\partial}{\partial u} + \frac{m+1}{m}w\frac{\partial}{\partial w} + \frac{1-m}{m}v\frac{\partial}{\partial v} \tag{3.4}$$

The collection of functionally independent invariants is given in this case by the relations

$$I_1 = x\exp\frac{-t}{k_2}, \qquad I_2 = \varphi\exp\frac{-t}{mk_2}, \qquad I_3 = u\exp\frac{-t}{mk_2}$$

$$I_4 = w\exp\frac{-(m+1)t}{mk_2}, \qquad I_5 = v\exp\frac{-(1-m)t}{mk_2}$$

The system S/H_3 can be reduced to

$$mk_2c_1{I_2}^{2m}\frac{d^2I_2}{d{I_1}^2} = I_2(1+\alpha mk_2) - \alpha mk_2I_3 - mI_1\frac{dI_2}{dI_1} - 2m^2k_2c_1I_2^{2m-1}\left(\frac{dI_2}{dI_1}\right)^2$$

$$mk_2c_1{I_2}^{2m}\frac{d^2I_3}{d{I_1}^2} = I_3(\alpha mk_2 - 1) + mI_1\frac{dI_3}{dI_1}$$

$$I_4 = c_1{I_2}^{2m}\frac{dI_2}{dI_1}, \qquad I_5 = \frac{dI_3}{dI_1} \tag{3.5}$$

Among the boundary conditions suitable in this case are those of the (3.4) type.

The optimal control in the above cases is constructed after the appropriate invariants have been determined.

Because the coordinates ξ_t, ξ_x, ξ_u, ξ_φ do not depend on the ancillary variables w and v, we can write out a 'truncated' operator which refers to transformations in the space t, x, u, φ. We then consider the equations of the variational problem in the form (1.1), (1.6).

It was not our intention in the present paper to analyze the relationship between the boundary conditions of the problem with the transformation groups. We have merely established that each transformation group must be matched with boundary conditions of a certain type.

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